Problem 1. Assume that $f : (1, \infty) \to \mathbb{R}$ is continuous and bounded. Prove that there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} x_n = +\infty \quad \text{and} \quad \lim_{n\to\infty} \left(f\left(x_n + 2021\right) - f\left(x_n\right)\right) = 0$$

Solution.

Let $g(x) = f(x+2021) - f(x)$. Then there are two possibilities. (1) There exists $x_0 > 1$ such that $g(x)$ is positive (or negative) for all $x > x_0$. (2) There is no such x_0.

In case (1), if, for example, g is positive on (x_0, ∞), then the sequence $\{f\left(x_0 + 2021n\right)\}$ is monotonically increasing. Since f is bounded, the following limit exists and is finite:

$$\lim_{n\to\infty} f\left(x_0 + 2021n\right) = \lim_{n\to\infty} f\left(x_0 + 2021(n+1)\right) = \lim_{n\to\infty} f\left(x_0 + 2021n + 2021\right)$$

Therefore, one can take $x_n = x_0 + 2021n$.

In case (2), by the intermediate value property of g, for every positive integer $n > 1$ there is $x_n > n$ such that $g\left(x_n\right) = 0$

Problem 2. A natural number n is given. For which $k \in \{1, 2, \ldots, n\}$ does a square matrix A of order n with integer elements exist such that all minors of order k (that is, determinants of matrices obtained from A deleting $n-k$ rows and $n-k$ columns) are odd?

Solution.

Answer: For $k = 1, n-1, n$.

Examples:

i) $k = 1$ is a matrix of all 1;

ii) $k = n$ is the identity matrix;

iii) $k = n-1$ — add a row and a column to the identity matrix of order $n-1$ so that the sum in each row and in each column is even.

Suppose that such a matrix is found for $k \geqslant 2$ and $n \geqslant k+2$. Let's focus on the $k \times (k+2)$ submatrix. Let's denote its columns $f_1, \ldots, f_{k+2}$ and consider them as vectors over a field of two elements. Any k of them are linearly independent, because the corresponding determinant is equal to $1 (\mathrm{mod}\ 2)$ (i.e. odd). But any $k+1$ are linearly dependent, which means that the dependence is that the sum of these $k+1$ vectors is 0. Thus, each of the vectors f_{k+1}, f_{k+2} is equal to $f_1 + \ldots + f_k$. But this contradicts the linear independence of the vectors $f_3, \ldots, f_{k+2}$.

Problem 3. For any positive a, b prove the inequality

$$\ln\frac{(a+1)^2}{4a}\ln\frac{(b+1)^2}{4b} \geq \ln^2\frac{(a+1)(b+1)}{2(a+b)}.$$

Solution.

$$\ln\frac{(a+1)^2}{4a}\ln\frac{(b+1)^2}{4b} = \ln\left(1-\left(\frac{a-1}{a+1}\right)^2\right)\ln\left(1-\left(\frac{b-1}{b+1}\right)^2\right) =$$

$$\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{a-1}{a+1}\right)^{2n}\right)\times\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{b-1}{b+1}\right)^{2n}\right) \geqslant \left(\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{(a-1)(b-1)}{(a+1)(b+1)}\right)^{n}\right)^2$$

$$= \ln^2\left(1-\frac{(a-1)(b-1)}{(a+1)(b+1)}\right) = \ln^2\frac{(a+1)(b+1)}{2(a+b)}.$$

Problem 4. Let

$$a_n = 2\sum_{k=0}^{\infty} \frac{(-1)^{k(n+1)}}{(2k+1)^{n+1}}.$$

Prove that the sequence $\{a_n^{1/n}\}_{n=1}^{\infty}$ is strictly decreasing.

Solution.

For any $n \in \mathbb{N}$, prove the inequality $a_n^{1/n} > a_{n+1}^{1/(n+1)}$ which is equivalent to the following one

$$a_n^{n+1} > a_{n+1}^{n}.$$

Let n is odd. In this case, one has

$$a_n = 2\left(1 + \frac{1}{3^{n+1}} + \frac{1}{5^{n+1}} + \dots\right) > 2,$$
$$a_{n+1} = 2\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+2}} \le 2 \quad \text{(Leibnitz series)}.$$

Hence,

$$a_n^{n+1} > 2^{n+1} > 2^n \ge a_{n+1}^n.$$

Let n is even. Then

$$a_n = 2\left(\left(1 - \frac{1}{3^{n+1}}\right) + \left(\frac{1}{5^{n+1}} - \frac{1}{7^{n+1}}\right) + \dots\right) > 2\left(1 - \frac{1}{3^{n+1}}\right).$$

Using the Bernoulli inequality, one obtains

$$a_n^{n+1} > 2^{n+1}\left(1 - \frac{1}{3^{n+1}}\right)^{n+1} \ge 2^{n+1}\left(1 - \frac{n+1}{3^{n+1}}\right) \ge 2^{n+1}\cdot\frac{8}{9}.$$

One has

$$a_{n+1} = 2\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{n+2}} = 2\left(1 + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^{n+2}}\right).$$

Let us estimate the obtained series:

$$\sum_{k=1}^{\infty} \frac{1}{(2k+1)^{n+2}} \le \int_0^{+\infty} \frac{dt}{(2t+1)^{n+2}} = \frac{1}{2(n+1)}.$$

Consequently,

$$a_{n+1}^n \le 2^n\left(1 + \frac{1}{2(n+1)}\right)^n \le 2^n\sqrt{\left(1 + \frac{1}{2n}\right)^{2n}} < 2^n\sqrt{e} < 2^n\sqrt{3}.$$

Thus, it is sufficient to show that

$$2^{n+1} \cdot \frac{8}{9} > 2^n \sqrt{3}.$$

This inequality is equivalent to the following evident inequality

$$\frac{16}{9} > \sqrt{3} \quad \Leftrightarrow \quad 256 > 243.$$

Thus, we proved the inequality for even n also.

Problem 5. Evaluate the integral

$$\int\limits_{1}^{e-1} \lim_{n\to+\infty} \underbrace{\log(x+\log(x+\log(x+\ldots\log(x+\log}_{n \text{ times}} x)\ldots)))\,dx.$$

Here $\log(x) = \ln(x)$.

Solution.

Let's consider the sequence

$$y_n(x) = \underbrace{\log(x+\log(x+\log(x+\ldots\log(x+\log}_{n \text{ times}} x)\ldots))), \quad 1 \leqslant x \leqslant 1-e, \quad n = 1,2,3,\ldots,$$

all elements of which are defined correctly due to the fact that $\log t \in [0;+\infty)$ for any $t \in [1;+\infty)$ and take values from $[0;+\infty)$ in view of this fact. Since the sequence $y_n(x)$ is monotonically increasing,

$$y_n(x) = \underbrace{\log(x+\log(x+\log(x+\ldots\log(x+\log}_{n \text{ times}} x)\ldots))) \leqslant$$

$$\leqslant \underbrace{\log(x+\log(x+\log(x+\ldots\log(x+\log}_{n \text{ times}}(x+\log x))\ldots))) = y_{n+1}(x),$$

$$\forall x \in [1;e-1], \qquad n = 1,2,3,\ldots,$$

and is bounded from above,

$$y_n(x) = \underbrace{\log(x+\log(x+\log(x+\ldots\log(x+\log}_{n \text{ times}} x)\ldots))) \leqslant$$

$$\leqslant \underbrace{\log(e-1+\log(e-1+\log(e-1+\ldots\log(e-1+\log}_{n \text{ times}} e)\ldots))) = 1,$$

$$\forall x \in [1;e-1], \qquad n = 1,2,3,\ldots,$$

so the function

$$y(x) = \lim_{n\to+\infty} \underbrace{\log(x+\log(x+\log(x+\ldots\log(x+\log}_{n \text{ times}} x)\ldots)))$$

is defined for any $x \in [1;e-1]$ by virtue of Weierstrass theorem, $y(x) \in [0;1]$ for any $x \in [1;e-1]$ and the following identity takes place,

$$y(x) = \lim_{n\to+\infty} y_n(x) = \lim_{n\to+\infty} \log(x+y_{n-1}(x)) = \log(x + \lim_{n\to+\infty} y_{n-1}(x)) =$$

$$= \log(x+y(x)) \quad \Rightarrow \quad y(x) = \log(x+y(x)), \qquad \forall x \in [1;e-1].$$

It follows from this identity that the function $y(x)$ takes any its value for one only value of the variable x,

$$x = e^{y(x)} - y(x).$$

Thus, the inverse function $x = x(y) \equiv e^y - y$ for the function $y = y(x)$, $1 \leqslant x \leqslant e-1$ is strictly increasing (since $x'(y) = e^y - 1 > 0$ for any $y \in (0;1])$, $x(0) = 1$ and $x(1) = e - 1$. Taking into account that for any continuous strictly increasing function $f : [a;b] \to [c;d]$, $a \geqslant 0$, $c \geqslant 0$, the following equality is geometrically obvious,

$$\int\limits_a^b f(x)\,dx + \int\limits_{f(a)}^{f(b)} f^{-1}(y)\,dy = bf(b) - af(a),$$

we have that

$$\int\limits_1^{e-1} y(x)\,dx = (e-1)\cdot 1 - 1\cdot 0 - \int\limits_0^1 (e^y - y)\,dy = e - 1 - \left(e^y - \frac{y^2}{2}\right)\Big|_0^1 = \frac{1}{2}.$$

Problem 6.

Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers satisfying $\lim_{n\to\infty} a_n = 0$. Find a continuous function $f : (0, +\infty) \to (0, +\infty)$ (or prove that it does not exist) such that for all $x > 0$

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{f_n(x)}{a_n} = +\infty$$

where f_n is defined by $f_1 = f$ and $f_n = f \circ f_{n-1}$ for $n \geq 2$.

Solution.

Let $d = (d_n)_{n=1}^{\infty}$ be a decreasing sequence of positive real numbers with $\lim_{n\to\infty} d_n = 0$ and satisfying $\lim_{n\to\infty} \frac{d_n}{a_n} = +\infty$ and $d_{n+1} \geq \frac{n}{n+1} d_n$ for all $n \in \mathbb{N}$. Then, by induction, we have $d_{n+m} \geq \frac{n}{n+m} d_n$. Let us define f as $f(d_n) = d_{n+1}$ for all $n \in \mathbb{N}$, f linear on $[d_{n+1}, d_n]$ and constant on $[d_1, +\infty)$. We show that f has the desired property. In fact, for any $x > 0$ we have $f_1(x) = f(x) \leq d_1$ and since f is increasing on $(0, d_1)$ we have (by induction) $f_n(x) \leq d_n \to 0$. Moreover, for every $x > 0$ there exists $m \in \mathbb{N}$ such that $x > d_m$. Then, again by monotonicity of f, $f_n(x) \geq d_{m+n}$, and therefore

$$\frac{f_n(x)}{a_n} \geq \frac{d_{m+n}}{a_n} = \frac{d_{m+n}}{d_n} \cdot \frac{d_n}{a_n} \geq \frac{n}{n+m} \cdot \frac{d_n}{a_n} \to +\infty.$$

It remains to construct a sequence d with the above properties. Let us denote $b_n = \sup\{a_k : \ k \geq n\}$. Then the sequence $(b_n)_{n=1}^{\infty}$ is nonincreasing, $b_n \geq a_n$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} b_n = \limsup_{n\to\infty} a_n = 0$. Further, we define $c_n = \sqrt{b_n} + \frac{1}{n}$. The sequence $(c_n)_{n=1}^{\infty}$ is decreasing, converges to zero and

$$\frac{c_n}{a_n} \geq \frac{\sqrt{b_n}}{b_n} = \frac{1}{\sqrt{b_n}} \to +\infty.$$

We define $d_1 = c_1$ and for $n \geq 2$

$$d_n = \max\left\{c_n, \frac{n-1}{n} d_{n-1}\right\}.$$

Then $d_n \geq c_n$, hence $\lim_{n\to\infty} \frac{d_n}{a_n} = +\infty$ and we have $d_{n+1} \geq \frac{n}{n+1} d_n$. We show that d is decreasing with limit zero. We either have $d_n = \frac{n-1}{n} d_{n-1} < d_{n-1}$ or $d_n = c_n < c_{n-1} \leq d_{n-1}$. Hence, d is decreasing. Further, we either have a subsequence d_{n_k} where $d_{n_k} = c_{n_k}$. Then $\lim_{k\to\infty} d_{n_k} = \lim_{k\to\infty} c_{n_k} = 0$ and due to monotonicity $\lim_{n\to\infty} d_n = 0$. Or, there exists n_0 such that for all $n \geq n_0$ we have $d_n = \frac{n-1}{n} d_{n-1}$. Hence, $d_n = \frac{n_0}{n} d_{n_0} \to 0$.

Problem 7.

Has the equation $y'' + x^2 (y')^3 (2 + \sin(x - y)) = 0$ a non-constant solution defined in a neighborhood of ∞ and having a finite non-zero limit as $x \to \infty$?

Solution.

We will prove the existence of a strictly monotonic solution with a finite non-zero limit as $x \to \infty$. Let $z(t)$ be the inverse function of such a solution: $z(y(x)) = x, \;\; t = y(z(t))$. Then

$$
\begin{aligned}
1 &= y'(z(t)) \cdot \dot{z}(t), \\
0 &= y''(z(t)) \cdot \dot{z}(t)^2 + y'(z(t)) \cdot \ddot{z}(t), \\
0 &= -\frac{z(t)^2}{\dot{z}(t)^3}(2 + \sin(z(t) - y(z(t)))) \cdot \dot{z}(t)^2 + \frac{\ddot{z}(t)}{\dot{z}(t)}, \\
0 &= -z(t)^2(2 + \sin(z(t) - t)) + \ddot{z}(t),
\end{aligned}
$$

whence $\ddot{z} = (2 + \sin(z - t))\, z^2 = p(t, z)z^2$ with the Lipschitz continuous function $p(t, z)$ satisfying $1 \le p(t, z) \le 3$.

Consider the maximally extended solution to the last equation with the initial data $z(0) = 7, \;\; \dot{z}(0) = 13$.

Consider also the solution $w(t) = 6(t - 1)^{-2}, \;\; t < 1$, to the equation $\ddot{w} = w^2$ and note that $w(0) = 6 < 7, \;\; \dot{w}(0) = 12 < 13$.

Both z and w are positive and strictly increasing for $t > 0$ in their domains.

Moreover, for these t we have $w(t) < z(t)$ and $\dot{w}(t) < \dot{z}(t)$. Indeed, if not so, then let $t_1 > 0$ be the minimal t breaking one of these inequalities. Then they are satisfied on $[0; t_1)$ and

$$
\begin{aligned}
z(t_1) &= 7 + \int_0^{t_1} z(t)\, dt > 6 + \int_0^{t_1} w(t)\, dt = w(t_1), \\
\dot{z}(t_1) &= 13 + \int_0^{t_1} p(t, z(t))z(t)^2\, dt > 12 + \int_0^{t_1} w(t)^2\, dt = \dot{w}(t_1),
\end{aligned}
$$

which contradicts to the choice of t_1.

Note that $w(t) \to \infty$ as $t \to 1$. Hence either the same is for $z(t)$ or the right boundary point $t_* > 0$ of the domain of z is less than 1.

If $\lim_{t \to t_*} z(t) < \infty$, then $\ddot{z} = p(t, z(t))z(t)^2$ is bounded and therefore $\dot{z}$ also has a finite limit, which makes $z(t)$ extensible to the right of t_*. So, $z(t) \to \infty$ as $t \to t_*$. The inverse function of $z(t)$ can be defined at least on $[7, \infty)$ and tends to $t_* \in (0; 1]$ at infinity.